THE DOUBLE SHUFFLE RELATIONS FOR p-ADIC MULTIPLE ZETA VALUES

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Abstract. We give a proof of double shuffle relations for p-adic multiple zeta values by developing higher dimensional version of tangential base points and discussing a relationship with two (and one) variable p-adic multiple polylogarithms.

Contents

0. Introduction

In this paper we will prove a set of formulas, known as double shuffle relations, relating the p-adic multiple zeta values defined by the second named author in [F1]. These formulas are analogues of formulas for the usual (complex) multiple zeta values. These have a very simple proof which unfortunately does not translate directly to the p-adic world.

Recall that the (complex) multiple zeta value $\zeta(\mathbf{k})$, where $\mathbf{k}$ stands for the multi-index $\mathbf{k}=(k_1,\dots,k_m)$, is defined by the formula

$$\zeta(\mathbf{k})=\sum_{\substack{0<n_1<\cdots<n_m\\ n_i\in\mathbf{N}}}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}}\ ,\tag{0.1}$$

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The series is easily seen to be convergent assuming that $k_m > 1$.

Multiple zeta values satisfy two types of so called shuffle product formulas, expressing a product of multiple zeta values as a linear combination of other such values. The first type of formulas are called series shuffle product formulas (sometimes called by harmonic product formulas). The simplest example is the relation

$$\zeta(k_1)\cdot\zeta(k_2) = \zeta(k_1,k_2) + \zeta(k_2,k_1) + \zeta(k_1+k_2)\ , \tag{0.2}$$

which is easily obtained from the expression (0.1) by noting that the left hand side is a summation over an infinite square of pairs (n_1, n_2) of the summand in (0.1), and that summing over the lower triangle (respectively the upper triangle, respectively the diagonal) gives the three terms on the right hand side. Every series shuffle product formula has this type of proof.

The second type of shuffle product formulas, known as iterated integral shuffle product formulas, is somewhat harder to establish and follows from the description of multiple zeta values in terms of multiple polylogarithms. More precisely. The one variable multiple polylogarithm is defined by the formula

$$\mathrm{Li}_{\mathbf{k}}(z) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i\in\mathbf{N}}} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}\ , \tag{0.3}$$

near $z = 0$. It can then be extended as a multi-valued function to $\mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$. We clearly have the relation $\lim_{z\to 1} \mathrm{Li}_{\mathbf{k}}(z) = \zeta(\mathbf{k})$.

Multiple polylogarithms can be written using the theory of iterated integrals due to Chen [Ch]. In other words, they satisfy a system of unipotent differential equations. This gives an integral expression for multiple polylogarithms. By substituting $z = 1$ and splitting the domain of integration in the right way we obtain the iterated integral shuffle product formulas, a simple example of which is the formula

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{i=0}^{k_1-1} \binom{k_2-1+i}{i} \zeta(k_1-i, k_2+i) + \sum_{j=0}^{k_2-1} \binom{k_1-1+j}{j} \zeta(k_2-j, k_1+j). \tag{0.4}$$

In [F1] the second named author defined the p-adic version of multiple zeta values and studied some of their properties. The defining formula (0.1) can not be directly used p-adically because the defining series does not converge. Instead, one must use an indirect approach based on the theory of Coleman integration [Co, Bes1]. Coleman's theory defines p-adic analytic continuation for solutions of unipotent

differential equations "along Frobenius". Coleman used his theory initially to define p-adic polylogarithms. In [F1] Coleman integration was used to define one variable p-adic multiple polylogarithms. Taking the limit at 1 in the right way one obtains the definition of p-adic multiple zeta values. It is by no means trivial that the limit even exists or is independent of choices, and this is the main result of [F1].

Given their definition, it is not surprising that for p-adic multiple zeta values it is the iterated integral shuffle product formulas that are easier to obtain p-adically. In [F1] the series shuffle product formulas were not obtained. The purpose of this work is to prove (Theorem 7.1) these formulas, and as a consequence the double shuffle relations (Corollary 7.2) for p-adic multiple zeta values.

To prove the main theorem it is necessary to use the theory of Coleman integration in several variables developed by the first named author in [Bes1]. The reason for this is quite simple - If one tries to replace multiple zeta values by multiple polylogarithms in the proof of (0.2) sketched above one easily establishes the formula

$$\mathrm{Li}_{k_1}(z)\,\mathrm{Li}_{k_2}(w) = \mathrm{Li}_{k_1,k_2}(z,w) + \mathrm{Li}_{k_2,k_1}(w,z) + \mathrm{Li}_{k_1+k_2}(zw)\ , \tag{0.5}$$

which is a two variable formula. It seems impossible to obtain a one variable version of the same formula. The proof of the main theorem thus consists roughly speaking of showing that (0.5) extends to Coleman functions of several variables and then taking the limit at $(1,1)$.

Since taking the limit turned out to be rather involved in [F1], we opted for an alternative approach, which was motivated by a letter of Deligne to the second named author [D2]. Deligne observes that taking the limit at 1 for the multiple polylogarithm can be interpreted as doing analytic continuation from tangent vectors at 0 and 1, using the theory of the tangential basepoint at infinity introduced in [D1]. To analytically continue (0.5) and obtain the series shuffle product formula we analyze a more general notion of tangential basepoint sketched in loc. cit. and examine among other things its relation with Coleman integration.

To give a precise meaning of the limit value to $(1,1)$, we work over the moduli space $\mathcal{M}_{0,5}$ of curves of $(0,5)$-type and the normal bundles for the divisors at infinity $\overline{\mathcal{M}_{0,5}} - \mathcal{M}_{0,5}$ ($\overline{\mathcal{M}_{0,5}}$: a compactification of $\mathcal{M}_{0,5}$). Two variable p-adic multiple polylogarithms are introduced. They are Coleman functions over $\mathcal{M}_{0,5}$. In §6 their analytic continuation to the normal bundle will be discussed. In particular, we will relate the behavior of the analytic continuation of two variable multiple

polylogarithm to a normal bundle with one variable multiple polylogarithm and then we get p-adic multiple zeta values as "special values" of two variable multiple polylogarithms.

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1. Tannakian duality and fundamental groups

This section is introductory, and contains some basics of the theory of Tannakian categories (for which the standard reference is [DM]) and some ways it can be used to construct fundamental groups and path spaces in algebraic geometry.

The theory of Tannakian categories has its roots in Tannaka-Krein duality, itself a generalization of Pontrjagin duality, which shows how to reconstruct a compact topological group out of the category of its representations. Subsequently, it was realized by Grothendieck that the reconstruction can be done, in the somewhat different context of affine group schemes, based on some formal properties of this category of representations. Consequently, any category with some additional structures and properties, formalized in the notion of a Tannakian category, gives rise to an affine group scheme.

A Tannakian category (or more precisely a neutral Tannakian category) over a field F is a rigid abelian F-linear tensor category possessing a fiber functor [DM, Definition 2.19]. An abelian F-linear category $\mathcal{T}$ becomes a tensor category with the addition of a bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ satisfying certain axioms mimicking those of the tensor product of vector spaces. It is rigid if it satisfies [DM, Definition 1.7]. A tensor functor $G : \mathcal{T} \to \mathcal{T}'$ is a functor together with a functorial isomorphism $G(X) \otimes G(Y) \to G(X \otimes Y)$ satisfying a few obvious properties [DM, Definition 1.8]

The category Vec_F of finite dimensional vector spaces over F is a rigid abelian F-linear tensor category. The final data in the definition of a Tannakian category is a fiber functor, which by definition is an exact F-linear faithful tensor functor $\omega : \mathcal{T} \to \mathrm{Vec}_F$.

Suppose now that G is an affine group scheme over F. One than has the category $\mathrm{Rep}_F(G)$ of finite dimensional F-rational representations of G. This is clearly a Tannakian category with fiber functor sending a representation to its underlying K-vector space. The main result of Tannakian duality [DM, Theorem 2.11] says that $\mathrm{Rep}_F(G)$ determines G uniquely, and conversely, any Tannakian category arises as $\mathrm{Rep}_F(G)$ for some G.

Recall that an affine group scheme over F can be defined as a group valued functor on the category of F-algebras which is represented by an F-algebra. The way that $\mathrm{Rep}_F(G)$ determines G is as follows: G is isomorphic to the group valued functor $\mathrm{Aut}^{\otimes}(\omega)$ whose value on an F-algebra R is

$$\mathrm{Aut}^{\otimes}(\omega)(R) := \mathrm{Hom}(\omega \otimes R, \omega \otimes R) \ ,$$

where $\omega \otimes R$ is the composition of ω with the change of ring functor and Hom is in the category of tensor functors. We will often write $\pi_1(\mathcal{T}, \omega)$ instead of $\mathrm{Aut}^{\otimes}(\omega)$ to indicate the dependency on the category $\mathcal{T}$.

Tannakian duality can be used to define all kinds of “fundamental groups” in algebraic geometry and algebraic topology. The basic example to keep in mind is that of the category Loc_X of local system of finite dimensional $\mathbf{Q}$-vector spaces on a connected topological space X. Given a choice of a point $x \in X$, this is equivalent to the category of finite dimensional $\mathbf{Q}$-representations of the fundamental group $\pi_1(X, x)$. It is clearly a Tannakian category over $\mathbf{Q}$. Each point $x \in X$ determines a fiber functor ω_x where $\omega_x(V) =$ fiber of V at x. The obvious map $\pi_1(X, x) \to \pi_1(\mathrm{Loc}_X, \omega_x)(\mathbf{Q})$ is explicitly given by

$$\gamma \mapsto (V \mapsto \text{monodromy along the loop } \gamma \in \mathrm{Aut}(\omega_x(V))) \ . \tag{1.1}$$

We may also use Tannakian duality to define generalized path spaces. Namely, if $\mathcal{T}$ is a Tannakian category and ω_1 and ω_2 are two fiber functors on $\mathcal{T}$. Then there is a affine scheme $\pi_1(\mathcal{T}, \omega_1, \omega_2) = \mathrm{Hom}^{\otimes}(\omega_1, \omega_2)$ whose points in an F-algebra R are

$$\mathrm{Hom}^{\otimes}(\omega_1, \omega_2)(R) := \mathrm{Hom}(\omega_1 \otimes R, \omega_2 \otimes R) \ ,$$

Again Hom taken in the category of tensor functors. There is an obvious composition map,

$$\circ : \pi_1(\mathcal{T}, \omega_2, \omega_3) \times \pi_1(\mathcal{T}, \omega_1, \omega_2) \to \pi_1(\mathcal{T}, \omega_1, \omega_3) \ , \tag{1.2}$$

and in particular $\pi_1(\mathcal{T}, \omega_1, \omega_2)$ is a homogeneous space for an action from the left of $\pi_1(\mathcal{T}, \omega_1)$ as well as for an action from the right of $\pi_1(\mathcal{T}, \omega_2)$ and is principal for both actions. For $\mathcal{T} = \mathrm{Loc}_X$ we have a

map $\pi_1(X,x,y) \to \pi_1(\mathrm{Loc}_X, \omega_x, \omega_y)(\mathbf{Q})$, where $\pi_1(X,x,y)$ is the space of paths between x and y on X up to homotopy, given by

$$\gamma \mapsto (V \mapsto \text{parallel transport along the path } \gamma \in \mathrm{Hom}(\omega_x(V), \omega_y(V))) \ . \tag{1.3}$$

A tensor functor $F : \mathcal{S} \to \mathcal{T}$ clearly induces a map of path spaces

$$F^* : \pi_1(\mathcal{T}, \omega_1, \omega_2) \to \pi_1(\mathcal{S}, \omega_1 \circ F, \omega_2 \circ F) \tag{1.4}$$

which is explicitly given by sending $\gamma \in \mathrm{Hom}^{\otimes}(\omega_1 \otimes R, \omega_2 \otimes R)$ to $F^*(\gamma)$ given by $F^*(\gamma)_S = \gamma_{F(S)}$. This is clearly compatible with composition. In the case of categories of local systems, when F is induced by the continuous map $f : X \to Y$ it is easy to see that this map is compatible, via (1.3) with the usual map induced by f on path spaces.

2. Coleman's p-adic integration

In this section we recall the theory of Coleman's p-adic integration in several variables introduced in [Bes1] and make some additional constructions which are required for the current work. In some respects, the similar theory of Vologodsky [V] is more suitable for the current setting. However, because it is much less explicit it seems harder to adapt it to the tangential constructions of §3.

Recall from [Bes1] that the basic setup for Coleman integration theory is as follows: We have a field of characteristic p, κ, which we assume for simplicity to be algebraically closed and a discrete valuation ring $\mathcal{V}$ with fraction field K and residue field κ. A rigid triple is a triple $T = (X, Y, P)$ consisting of P, a formal p-adic $\mathcal{V}$-scheme, Y a closed κ-subscheme of P which is proper over $\mathrm{Spec}(\kappa)$ and X an open κ-subscheme of Y such that P is smooth in a neighborhood of X.

A simple case where rigid triples arise (see [Bes1, Definition 5.1]) is that of tight rigid triples. Let $\mathcal{X} \subset \mathcal{Y}$ be an open immersion of $\mathcal{V}$-schemes such that $\mathcal{X}$ is smooth and $\mathcal{Y}$ is complete. The associated triple $T_{(\mathcal{X},\mathcal{Y})} := (\mathcal{X} \otimes_{\mathcal{V}} \kappa, \mathcal{Y} \otimes_{\mathcal{V}} \kappa, \hat{\mathcal{Y}})$, where $\hat{\mathcal{Y}}$ is the p-adic completion of $\mathcal{Y}$ is called a tight rigid triple. An affine rigid triple is a tight rigid triple $T_{(\mathcal{X},\mathcal{Y})}$ with $\mathcal{X}$ affine.

Given a rigid triple as above we have the category of unipotent overconvergent isocrystals on T, denoted $\mathcal{U}n(T)$. This is the category of sheaves on the tube of X in P, overconverging into the tube of Y in P in the sense of Berthelot [Ber], with an integrable connection. This category depends, up to a unique isomorphism, only on X, so we will often denote it simply by $\mathcal{U}n(X)$.

A Frobenius endomorphism of X is a κ-linear endomorphism of X, which is some power of the geometric Frobenius. A Frobenius endomorphism ϕ induces an auto-functor ϕ^* on $\mathcal{U}n(X)$. Suppose that $T = T_{(\mathcal{X},\mathcal{Y})}$ is a tight rigid triple and that ϕ is the reduction of an endomorphism φ of the pair $(\mathcal{X},\mathcal{Y})$. Then the functor ϕ^* is given by the obvious functor induced by φ on $\mathcal{U}n(T)$.

Definition 2.1. A Frobenius fiber functor on $\mathcal{U}n(X)$ is a fiber functor ω together with an isomorphism of fiber functors (which we write as an equality to simplify matters) $\omega \circ \phi^* = \omega$

For two Frobenius fiber functors ω_1 and ω_2 there is an obvious action of ϕ^* on the path space $\pi_1(\mathcal{U}n(X), \omega_1, \omega_2)$ coming from the action (1.4). The main result of [Bes1] (Corollary 3.2) can be stated, in a slightly generalized form, as the following:

Theorem 2.2. *Given any two Frobenius fiber functors on $\mathcal{U}n(X)$, ω_1 and ω_2, there exists a canonical invariant path, i.e., an isomorphism of fiber functors, $a_{\omega_1,\omega_2} : \omega_1 \to \omega_2$, which is fixed by ϕ^*. We will call the isomorphism $\omega_1 \to \omega_2$ analytic continuation along Frobenius. It has the basic compatibility property that $a_{\omega_2,\omega_3} \circ a_{\omega_1,\omega_2} = a_{\omega_1,\omega_3}$.*

Proof. We only sketch the main idea, which is rather simple, and refer to reader to loc. cit. for the full story. The key point is that for any Frobenius fiber functor ω there is an action ϕ^* on the fundamental group $\pi_1(\mathcal{U}n(X))$, which is a pro-unipotent group, and its Lie algebra is controlled by negative tensor powers of $H^1_{\text{rig}}(X/K)$. The action of ϕ^* on $H^1_{\text{rig}}(X/K)$ has strictly positive weights, and consequently one can show that the map

$$g \mapsto \phi^*(g)g^{-1} \ , \ \pi_1(\mathcal{U}n(X), \omega) \to \pi_1(\mathcal{U}n(X), \omega) \ ,$$

is a bijection. Now, the space $\pi_1(\mathcal{U}n(X), \omega_1, \omega_2)$ is a $\pi_1(\mathcal{U}n(X), \omega_1)$-principal space and its ϕ^* action is compatible with the group action. From this it is very easy to deduce that there exists a unique ϕ^*-invariant element. The uniqueness immediately implies the compatibility. □

Usually, Frobenius fiber functors as above are obtained from geometric points $x : \operatorname{Spec} \kappa \to X$ by pullback, $\omega_x = x^*$, assuming that x is fixed by ϕ. Concretely, for an overconvergent isocrystal (M, ∇) the fiber functor $\omega_x(M, \nabla)$ is realized as the vector space of horizontal sections of ∇ on the residue disc

$$U_x =]x[_P \text{ (=points reducing to } x)$$

which is the tube in the sense of Berthelot. Since each point of X will be fixed by some Frobenius automorphism it is not hard to see that the invariant path is independent of the choice of the Frobenius endomorphism.

An abstract Coleman function on T is defined to be a triple (M, s, h) where

- $M = (M, \nabla)$ is a unipotent isocrystal on T.
- $s \in \mathrm{Hom}(M, \mathcal{O})$.
- h is a collection of sections, $\{h_x \in M(U_x),\ x \in X\}$, with $\nabla(h_x) = 0$, which correspond to each other via analytic continuation along Frobenius.

One can evaluate an abstract Coleman function on each residue disc U_x by taking $s(h_x)$. Coleman functions are then equivalence classes of abstract Coleman functions under a relation which essentially guarantees that their evaluations will be identical. In fact, a Coleman function is a connected component of the category of abstract Coleman functions, where a morphism $(M_1, s_1, h_1) \to (M_2, s_2, h_2)$ is a horizontal map $u : M_1 \to M_2$ such that $s_2 \circ u = s_1$ and $u(h_1) = h_2$. It is shown in [Bes1, Corollary 4.13] that two Coleman functions which coincide on an open subset of the tubular neighborhood $]X[_P$ of X in P coincide everywhere.

Remark 2.3. It will be important to note how the theory can be used to define iterated integrals. For a similar and more detailed discussion see the proof of Theorem 4.15 in [Bes1]. Suppose that we have a collection of Coleman functions F_i, arising from the abstract Coleman functions (M_i, s_i, h_i), and one-forms ω_i for $i = 1, \dots, k$. Suppose now that the locally analytic differential form $\omega = \sum F_i \omega_i$ is closed, and that we wish to find a Coleman function F such that $dF = \omega$. We may begin by constructing the connection $M = (\oplus M_i) \oplus \emptyset_T$ with the connection given by the formula

$$\nabla(m_1, \cdots, m_k, f) = \left(\nabla_1 m_1, \cdots, \nabla_k m_k, df - \sum \omega_i s_i(m_i)\right) .$$

This is clearly a unipotent connection. If it is integrable we immediately obtain our required Coleman function represented by the abstract Coleman function (M, s, y), where s is the projection on the last factor and h is obtained by choosing one residue disc U, extending the sum of the h_i to a horizontal section of M on U, and then analytically continuing along Frobenius. In general, ∇ will not be integrable. However, there is a maximal integrable subconnection $(M^{\mathrm{int}}, \nabla)$ of (M, ∇) [Bes1, (2.3)]. The fact that $\sum F_i \omega_i$ is closed guarantees that we may

choose h to be in $M^{\text{int}}(U)$ hence we obtain our Coleman function from (M^{int}, s, h).

One can apply the theory to other fiber functors. A simple case was already discussed in [Bes1, Section 5]. Here we will need an easy generalization (to the two dimensional case for simplicity) as follows: Suppose that $T = (X, Y, P) = T_{(\mathcal{X},\mathcal{Y})}$ is a rigid triple and that $y \in D = Y - X$ is a closed point and that D is locally given near y by the reduction of two parameters t_1 and t_2 on Y. Let A be the ring of Laurent series with coefficients in K,

$$A := \left\{ f(t_1, t_2) = \sum_{(i,j)\in \mathbf{Z}^2} a_{ij} t_1^i t_2^j \;\middle|\; f \text{ converges on } r < |t_1|, |t_2| < 1 \text{ for some } r \right\}$$

It follows from [Ber] that overconvergent isocrystals on T give rise to a connection on a (free) A-module.

Proposition 2.4. *An overconvergent unipotent connection on T has a full set of solutions in $B := A[\log(t_1), \log(t_2)]$.*

Proof. Overconvergent unipotent crystals are solved by iterated integration and the ring B is easily seen to be closed under partial integration with respect to t_1 and t_2. ☐

Note that the ring B is independent of choice of parameters t_1 and t_2. The following operation will be required later.

Definition 2.5. The constant term with respect to the parameters t_1 and t_2 of $f \in B$ is defined as follows: Let $f = \sum f_{ij} \log^i(t_1) \log^j(t_2)$. Let $f_{00} = \sum a_{ij} t_1^i t_2^j$. Then the constant term is a_{00}

Note that the constant term does depend on the choice of parameters.

Definition 2.6. The fiber functor ω_y of $\mathcal{U}n(X)$ associates to a unipotent overconvergent isocrystal the vector space of solutions in B.

Suppose that we have an endomorphism φ of the pair $(\mathcal{X}, \mathcal{Y})$ reducing to a Frobenius endomorphism fixing y (a sufficiently high power of a Frobenius endomorphism will fix y). Then, pulling back by φ gives an isomorphism $\omega_y \circ \varphi \cong \omega_y$ making it a Frobenius fiber functor in the sens of Definition 2.1. It is therefore possible to analytically continue along Frobenius to or from ω_y.

To end this section we note that by the naturality of the analytic continuation along Frobenius, it clearly extends to pro-unipotent isocrystals. We also note that as long as the underlying connections are defined over a discretely valued subfield it is possible to work over $\mathbf{C}_p$.

3. Tangential basepoints

In this section we first recall the theory of the tangential basepoint in the de Rham setting due to Deligne [D1]. In its simplest form this theory allows one to define fiber functors for the category of integrable connections on a curve associated with a tangent point "at infinity". We explain Deligne's interpretation of constant terms in terms of these tangential basepoints. In loc. cit. a higher dimensional theory is also sketched with no details, but these can easily be filled in, as we will do. We then discuss the possibility of analytically continuing solutions of a unipotent integrable connection along Frobenius a la Coleman to such fiber functors. We finally explain how one can iterate the construction of tangential basepoints in two different ways and get the same result and we close with some applications to Coleman integration theory.

We first recall the construction of Deligne [D1, 15. Theorie algébrique]. Let C be a curve over a field K of characteristic 0, smooth at a point P, and let t be a parameter at P (Deligne immediately passes to the completion at P but we will not do the same). Suppose (M, ∇) is a connection on $C - \{P\}$ with logarithmic singularities along P. This means that locally near P the connection ∇ can be written as $\nabla = d + \Gamma$, where Γ is a section of $\mathrm{End}(M) \otimes \Omega^1_C(\log P)$ with at most a simple pole at P. The parameter t induces naturally a parameter $\bar{t}$ on the tangent space $T_P(C)$ (the linear parameter taking the value 1 at the derivation d/dt). We associate with this data the connection on $T_P(C) - 0$ on the trivial bundle with fiber M_P (fiber of M at P), given by $\mathrm{Res}_P(\nabla) := d + (\mathrm{Res}_P \Gamma)\,\mathrm{dlog}(\bar{t})$. This clearly defines a functor Res_P.

An easy computation shows that the functor Res_P does not depend on the parameter t. Deligne gives an alternative, coordinate free description of the same construction, making this fact evident. The valuation v_P on the fraction field $K(C)$ gives an algebra filtration F_P on this field and there is an obvious canonical isomorphism between the associated graded algebra $\mathrm{Gr}_P K(C)$ and the coordinate ring of $T_P(C) - 0$, given by sending a cotangent vector $d_P f$, thought of as a linear function on $T_P(C)$, to the image of f in $\mathrm{Gr}^1_P K(C)$.

Let $\Omega^1_{\mathcal{O}_C/K}(\log P)$ be the sheaf of differential forms with log singularities along P. We give $\Omega^1_{K(C)/K} = K(C) \otimes_{\mathcal{O}_C} \Omega^1_{\mathcal{O}_C/K}(\log P)$ the filtration induced from the filtration on the first term. The filtrations on $K(C)$ and on $\Omega^1_{K(C)/K}$ induces a filtration on $M \otimes_{\mathcal{O}_C} K(C)$ and on $M \otimes_{\mathcal{O}_C} \Omega^1_{K(C)/K}$ and the assumption that ∇ has log singularities implies that it preserves the filtration. It is easy to see that the induced connection on the associated graded is exactly $\mathrm{Res}_P(\nabla)$

The construction of Res_P gives us the option of producing more fiber functors for the category of connections on C by taking the fiber at a point of $T_P(C) - 0$. A particular case of this construction gives the notion of a constant term for a horizontal section of this connection as we now explain. Suppose that the connection ∇ is unipotent. Then it is very easy to see that one can find a basis of formal horizontal sections to ∇ near P with coefficients in the ring $K[[t]][\log(t)]$, where $\log(t)$ is treated as a formal variable whose derivative is dt/t. Let v be such a horizontal section. There is a sense in which we can specialize v to the fiber M_P, namely, taking the constant term.

Definition 3.1. The constant term of v is obtained by formally setting $t = \log(t) = 0$.

The justification for this definition is that over the complex numbers one has $\lim_{t\to 0} t\log(t) = 0$. Thus in situations where the solutions has coefficients in $K[[t]] + tK[[t]][\log(t)]$ this is indeed the constant term. The same can be argued p-adically, provided one takes the appropriate notion of limit. It is important to note that the situation described above is indeed what happened for p-adic multiple polylogarithms near 1, by the main result of [F1].

Similarly, there is a basis of (this time global) solutions to $\mathrm{Res}_P \nabla$ with coefficients in $K[\log(\bar{t})]$. In fact, all the solutions are of the form $\exp(\mathrm{Res}_P \Gamma \cdot \log \bar{t}) \cdot v$ with $v \in M_P$ and the exponential is a finite sum as $\mathrm{Res}_P \Gamma$ is nilpotent. In this case we can again take the constant term. This can now be interpreted as simply evaluating at 1 with the convention that $\log(1) = 0$. Thus, we may formally interpret taking the constant term as continuing to the tangential vector 1 at P. To make this more than a mere heuristic, though, one needs to introduce a topology. It can be made precise in the complex case [D1] and we will show this also in the p-adic case (see below Proposition 4.5).

The higher dimensional generalization is now fairly clear. Suppose that a smooth variety X is given and in it a divisor $D = \sum_{i\in I} D_i$ with normal crossings. We assume that all the components D_i are smooth. For a subset $J \subset I$ we set $D_J = \cap_{j\in J} D_j$. Let N_J be the normal bundle to D_J and let N_J^0 be the complement in N_J of $N_{J'}|_{D_J}$ for $J' \subset J$. Note that one only need to take J' smaller by one index and that $N_\emptyset$ is considered as the zero section. Thus, for example, N_j^0 is the (one-dimensional) normal space to D_j minus the zero section. Finally, we denote by N_J^{00} the restriction of N_J^0 to $D_J^0 := D_J - \cup_{j\notin J} D_j$. Following Deligne we construct, given a connection on X with logarithmic singularities along D, a connection on every N_J^{00} with logarithmic singularities "at infinity". Infinity here means the union of the hyperplane

at infinity for the normal bundle N_J, the hyperplanes $N_J - N_J^0$, and $N_J|_{D_J - D_J^0}$.

For each $j \in J$ consider the valuation v_j on $K(X)$ associated with the divisor D_j. Let $\mathcal{O}_X(D^{-1})$ be the localization of $\mathcal{O}_X$ at D. There exists a multi-filtration F_J on $\mathcal{O}_X(D^{-1})$, indexed by tuples $\chi = (\chi_j)_{j\in J}$, such that F_J^χ is the $\mathcal{O}_X$-module generated by $\{f \in \mathcal{O}_X(D^{-1}),\ v_j(f) \geq \chi_j$ for all $j \in J\}$. It is easy to see that $\operatorname{Spec}(\operatorname{Gr}_J \mathcal{O}_X(D^{-1}))$ is precisely N_J^{00}. Again, a natural map sends a differential form df, viewed as a function on N_J^{00}, linear in the bundle direction, to the image of f in $\operatorname{Gr}_J^1 \mathcal{O}_X(D^{-1})$.

Remark 3.2. Let $\operatorname{Spec}(A)$ be an affine patch on X with coordinates $t_i \in A$, $i \in I$, such that $D_i \cap \operatorname{Spec}(A)$ is defined by $t_i = 0$. Let $B = A/(t_j, j \in J)$, such that $D_J \cap \operatorname{Spec}(A) = \operatorname{Spec}(B)$ Let $t^\chi := \prod_{j\in J} t_j^{\chi_j}$. Then $F_J^\chi = At^\chi$ and it follows easily that $\operatorname{Gr}_J \cong B[t_j^\pm, j \notin J]$

Now suppose we have a connection with logarithmic singularities along D, $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega_X^1(\log D)$. We give $\Omega_X^1(D^{-1}) = \Omega_X^1(\log D) \otimes \mathcal{O}_X(D^{-1})$ the induced filtrations from the filtration on $\mathcal{O}_X(D^{-1})$. It is easy to see that the differential d preserves the filtration. Now let $M(D^{-1}) = M \otimes \mathcal{O}_X(D^{-1})$ and $M \otimes \Omega_X^1(D^{-1})$ have the induced filtrations. It follows that the extended connection $\nabla : M(D^{-1}) \to M \otimes \Omega_X^1(D^{-1})$ respects the filtration. The connection we are looking for is the *gr* of this connection. It is evident from this construction that if ∇ is flat, so will be the resulting connection.

Definition 3.3. We call the resulting connection the residue connection on N_J^{00} along D_J and denote it by $\operatorname{Res}_{D,J} \nabla$.

Note that the construction really depends on both D and J and not just on D_J. It is evident that $\operatorname{Res}_{D,J}$ is a tensor functor.

As a consequence of the construction above we obtain more fiber functors for connections.

Definition 3.4. If $x \in N_J^{00}$ we let $\omega_{x,J}$ be the fiber functor $\omega_{x,J} := \omega_x \circ \operatorname{Res}_{D,J}$.

Suppose now that $f : (X', D') \to (X, D)$ is a morphism (i.e., takes D' to D) and that $f^* D_j = n_j D_j'$. We get an induced map

$$N_f : N_{J'}^{00} \to N_J^{00} ,$$

by looking at the map on the graded sheaf of rings, obtained from f, sending Gr_J^χ to $\operatorname{Gr}_{J'}^{\overline{n}\cdot\chi}$, where $\overline{n} \cdot \chi$ is the vector with components $n_j \chi_j$. The following Lemma is trivial.

Lemma 3.5. *Suppose that* Fr *is the absolute Frobenius. Then* N_{Fr} *is also the absolute Frobenius.*

The constructions above can be iterated, and they commute in a sense we will now explain. For simplicity, we now assume that the divisor D has only two components, D_1 and D_2. Consider N_1. It has on it a divisor D' with normal crossings consisting of D'_1 = the zero section and D'_2 = restriction of N_1 to $D_{12} = D_1 \cap D_2$. The complement of this divisor is exactly N_1^{00}. We have the following result.

Proposition 3.6. *Write* $N = N^{00}_{D'_1,D'_2}$ *be the construction above performed on* N_1 *with respect to* D'. *Then there exists a natural isomorphism* $\psi : N \xrightarrow{\sim} N_{12}^{00}$ *in such a way that for any connection* M *on* X *with logarithmic singularities along* D *we have a natural isomorphism* $\psi^* \operatorname{Res}_{D,12} M \to \operatorname{Res}_{D',12} \operatorname{Res}_{D,1} M$.

Proof. The isomorphism ψ is essentially the isomorphism, existing on any object with two filtrations, $\mathrm{Gr}_{12} \cong \mathrm{Gr}_2 \mathrm{Gr}_1$, applied to the filtrations we have. The isomorphism on the Res's is then a completely formal consequence of this. ☐

Remark 3.7. To make very concrete the situation of the last Proposition, and to make it clear how it is going to be used, consider the simplest situation, X is the affine plane $\mathbf{A}^2$ with coordinates x and y, $D = D_1 \cup D_2 = \{x = 0\} \cup \{y = 0\}$. Clearly, N_1, the tangent space to $\{x = 0\}$, is again isomorphic to $\mathbf{A}^2$, and we may consider on it the curve C which is the section $x = 1$. Proposition 3.6, together with the functoriality of the Res construction, allows us to get the following: Consider the fiber functor on connections on X with logarithmic singularities along D: First apply Res_1, then restrict to C, finally consider the fiber functor at the tangential point 1 to C at $y = 0$. Then this fiber functor is naturally isomorphic to the fiber functor at the normal vector $(1, 1)$ at the point $(0, 0)$.

4. Analytic continuation to tangential points

We now turn to the relation between the tangential theory described in the previous section and the analytic continuation along Frobenius explained in §2. Suppose then that K is a p-adic field with ring of integers $\emptyset_K$ and residue field κ. Let X be a smooth scheme over $\emptyset_K$ and let $D = \sum D_i$ be a divisor with relative normal crossings on X. We assume as before that the components D_i are smooth. The schemes N_J^{00} are then also smooth over $\emptyset_K$. Define the category $Un(X, D)$ of unipotent connections on X_K with logarithmic singularities along D_K.

This is a Tannakian category over K. Suppose that $x \in (X-D)(\kappa)$ and $y \in N_J^{00}(\kappa)$ are two points on the special fibers. Restricting $(M, \nabla) \in Un(X, D)$ to U_x and taking horizontal sections we obtain a fiber functor ω_x on $Un(X, D)$ as in §2. Similarly, restricting $\mathrm{Res}_{D_K,J} \nabla$ to the residue disk U_y of y in N_J^{00} and taking horizontal sections we obtain a fiber functor ω_y. Then we have the following result.

Theorem 4.1. *For any two points x and y, which are either in $(X - D)(\kappa)$ or in $N_J^{00}(\kappa)$ for some J, there exists a canonical isomorphism (analytic continuation along a Frobenius invariant path), $a_{x,y} : \omega_x \xrightarrow{\sim} \omega_y$, such that the following properties are satisfied.*

1. *These isomorphisms are compatible in the sense that $a_{y,z} \circ a_{x,y} = a_{x,z}$.*
2. *If $x, y \in (X-D)(\kappa)$ then $a_{x,y}$ is the isomorphism of Theorem 2.2 pulled back via $Un(X, D) \to \mathcal{U}n((X - D)_\kappa)$*
3. *If $x, y \in N_J^{00}(\kappa)$ (for the same J) then $a_{x,y}$ is the isomorphism of Theorem 2.2 pulled back via $\mathrm{Res}_{D_K,J} : Un(X, D) \to \mathcal{U}n((N_J^{00})_\kappa)$*

The Frobenius invariant path is functorial with respect to morphisms $(X', D') \to (X, D)$ in the obvious sense.

Proof. Since the nerve of an affine covering is contractible, we are easily reduced to the following case: $X = \mathrm{Spec}(A)$ is affine and D is defined on it by the vanishing of $t_1 \cdot t_2 \cdots t_k$, where t_i are parameters. Let $A^\dagger$ be the weak completion of A in the sense of [Put] and let $B^\dagger$ be the weak completion of $A[(t_1 \cdots t_k)^{-1}]$. We clearly have $A^\dagger \subset B^\dagger$. We may further assume that there exists a map, whose reduction is a Frobenius endomorphism, $\varphi : B^\dagger \to B^\dagger$ preserving $A^\dagger$ and sending t_i to t_i^q. We consider the de Rham complex of $A^\dagger$ with logarithmic singularities along D_K,

$$\mathcal{DR}(A_K^\dagger, D) : \ A_K^\dagger \xrightarrow{d} \sum \Omega^1(A_K^\dagger/K)\,\mathrm{dlog}(t_i) \xrightarrow{d} \cdots$$

and the de Rham complex $\mathcal{DR}(B_K^\dagger)$ of $B_K^\dagger$. Note that the cohomology of this last complex is the Monsky-Washnitzer = rigid cohomology $H_{\mathrm{rig}}((X - D)_\kappa)$. There is an obvious map

$$\mathcal{DR}(A_K^\dagger, D) \to \mathcal{DR}(B_K^\dagger) \tag{4.1}$$

and an action of φ^* on both complexes compatible with this map.

Proposition 4.2. *The map* (4.1) *is a quasi-isomorphism.*

Proof. This follows essentially from the arguments in [BC, Section 6] ∎

We only need the fact that the induced map on H^1 is an injection. This immediately implies the following.

Corollary 4.3. *The map $\varphi^* : H^1(\mathcal{DR}(A^\dagger, D)) \to H^1(\mathcal{DR}(A^\dagger, D))$ is independent of the choice of φ and has strictly positive weights,*

Proof. The same is true for $H^1_{\mathrm{rig}}((X-D)_\kappa)$ as in the proof of Theorem 2.2.

□

Define now the category $Un(A^\dagger, D)$ of unipotent integrable $A^\dagger$-connections with logarithmic singularities along D. There is an action of φ^* on this category, hence on its fundamental group and torsors. By the same argument as in the proof of Theorem 2.2 we have using Corollary 4.3, that there exists a unique invariant path between any two fiber functors on $Un(A^\dagger, D)$ and that this is independent of the choice of φ. We have such a fiber functor ω_x for $x \in (X-D)_\kappa$, and since there is an obvious residual functor $Un(A^\dagger, D) \to \mathcal{U}n((N_J^{00})_\kappa)$, defined in the same way as in Definition 3.3, we also have fiber functors ω_y for $y \in (N_J^{00})_\kappa)$. The fiber functors discussed before the theorem are the compositions of the ones we just defined with the restriction functor $Un(X, D) \to Un(A^\dagger, D)$ and we obtain our required path by pulling back along this restriction. With this definition, (1) and (2) are essentially clear, while (3) follows easily because by Lemma 3.5 we have that φ induces residually a morphism on N_J^{00} whose reduction is a Frobenius automorphism, and consequently the pullback via the residue functor of a Frobenius invariant path is φ-invariant. □

We can use the extension of Coleman functions described here to make a theory of Coleman functions "of algebraic origin" on the pair (X, D). The abstract Coleman functions are then triples (M, s, h) with $M \in Un(X, D)$ and $s : M \to \mathcal{O}_{X_K}$, while h is a compatible system of local horizontal sections as before. Theorem 4.1 allows us to define a map $\mathrm{Res}_{D,J}$ from these algebraic Coleman functions to Coleman functions on the normal bundle N_J^{00}. It is defined by

$$\mathrm{Res}_{D,J}(M, s, h) = (\mathrm{Res}_{D,J} M, \mathrm{Gr}\, s, h)$$

where h now refers to the induced compatible system of horizontal sections whose existence is guaranteed by Theorem 4.1.

In particular, we obtain from a Coleman function f as above a Coleman function $f^{(D)}$ on the normal bundle to D. The following propositions suggests how to compute it.

Proposition 4.4. *Suppose that we have the relation $df = \sum \omega_i g_i$ with f and g_i Coleman functions of algebraic origin. Then we have the*

relation $df^{(D)} = \sum(\operatorname{Res}_D \omega_i) g_i^{(D)}$*, where, if* ω *is locally written as* $\omega' + h \operatorname{dlog}(t)$*, with* t *the defining parameter for* D*, then* $\operatorname{Res}_D(\omega) = \omega'|_D + h|_D \operatorname{dlog}(\bar{t})$*.*

Proof. We consider the way f can be written as a Coleman function following Remark 2.3. In that description suppose that the connection M constructed there was indeed integrable. Then the result is easily obtained by computing the residual connection for M. The result remains true if M is not integrable. To see this we first note that the M^{int} construction is algebraic by Lemma 2.4 in [Bes2] and the discussion following it. One further sees that it is a connection with logarithmic singularities along D_K. It follows that we may obtain f from the abstract Coleman function (M^{int}, s, h) as in Remark 2.3. The residual construction is not limited to integrable connections and is functorial. Thus there is a horizontal map of connections $\operatorname{Res}_{D,i}(M^{\text{int}}) \to \operatorname{Res}_{D,i}(M)$. Consequently, $f^{(D)}$, which is defined on the normal bundle to D_i by $\operatorname{Res}_{D,i}(M^{\text{int}}, \operatorname{Gr} s, h)$, still satisfies the differential equation given by $\operatorname{Res}_{D,i}(M)$ □

We can now give, in the p-adic case, a tangential basepoint interpretation of the constant term.

Proposition 4.5. *In the situation of Definition 2.5 the constant term with respect to the parameters* t_1 *and* t_2 *of a horizontal section of a unipotent overconvergent isocrystal* ∇ *on* T *is the evaluation at the normal vector* $\bar{t}_1 = 1, \bar{t}_2 = 1$ *of the analytic continuation along Frobenius of this horizontal section.*

Proof. We only prove the 1-dimensional analogue as the proof is similar. In fact, we prove something stronger. Suppose (using the notation of this section) that the connection is locally given by $d + \Gamma$. Let f be a horizontal section for the connection ∇ and let v be its constant term. We claim that the analytic continuation of f is precisely the solution $c(f) := \exp(\operatorname{Res}_P \Gamma \cdot \log \bar{t}) \cdot v$ (the result is then proved by specializing to $\bar{t} = 1$). We already know that this is indeed a solution of $\operatorname{Res}_P(\nabla)$. The association $\nabla \mapsto (f \mapsto c(f))$ is clearly compatible with direct sums and tensor products hence defines a path. To show that this is a Frobenius invariant path we may assume that φ is such that $\varphi^*(t) = t^p$ The constant term of $\varphi^*(f)$ remains v while $\varphi^*\nabla$ is associated with $\varphi^*\Gamma$. It follows that

$$c(\varphi^* f) = \exp(\operatorname{Res}_P(\varphi^*\Gamma) \cdot \log \bar{t}) \cdot v = \exp(p \operatorname{Res}_P \Gamma \cdot \log \bar{t}) \cdot v = c(f)(\bar{t}^p) ,$$

which proves our claim. □

5. The set up of the moduli space $\mathcal{M}_{0,5}$

This section is devoted to giving a quick review on basic materials of the moduli space $\mathcal{M}_{0,5}$ of genus 0 curves with 5 distinct marked points and its Deligne-Mumford compactification $\overline{\mathcal{M}_{0,5}}$. The two (and one) variable p-adic multiple polylogarithms are introduced as Coleman functions on $\mathcal{M}_{0,5}$.

The moduli space $\mathcal{M}_{0,5}=\{(P_i)_{i=1}^5\in(\mathbf{P}^1)^5 | P_i\neq P_j\ (i\neq j)\}/PGL(2)$ is identified with

$$\{(x,y)\in\mathbf{A}^2\}\backslash\{x=0\}\cup\{y=0\}\cup\{x=1\}\cup\{y=1\}\cup\{xy=1\}. \tag{5.1}$$

This identification is given by sending (x,y) to 5 marked points in $\mathbf{P}^1$ given by $(0,x,1,\frac{1}{y},\infty)$. The symmetric group S_5 acts on $\mathcal{M}_{0,5}$ by $\sigma(P_i)=P_{\sigma^{-1}(i)}$ $(1\leqslant i\leqslant 5)$ for $\sigma\in S_5$. Especially for $c=(1,3,5,2,4)\in S_5$ its action is described by $x\mapsto\frac{1-y}{1-xy}$, $y\mapsto x$.

The Deligne-Mumford compactification of $\mathcal{M}_{0,5}$ is denoted by $\overline{\mathcal{M}_{0,5}}$. This space classifies stable curves of $(0,5)$-type and the above S_5-action extends to the action on $\overline{\mathcal{M}_{0,5}}$. This space is the blow-up of $(\mathbf{P}^1)^2(\supset\mathcal{M}_{0,5})$ at $(x,y)=(1,1)$, $(0,\infty)$ and $(\infty,0)$. The complement $\overline{\mathcal{M}_{0,5}}-\mathcal{M}_{0,5}$ is a divisor with 10 components: $\{x=0\}$, $\{y=0\}$, $\{x=1\}$, $\{y=1\}$, $\{xy=1\}$, $\{x=\infty\}$, $\{y=\infty\}$ and 3 exceptional divisors obtained by blowing up at $(x,y)=(1,1)$, $(\infty,0)$ and $(0,\infty)$. In particular for our convenience we denote $\{y=0\}$, $\{x=1\}$, the exceptional divisor at $(1,1)$, $\{y=1\}$ and $\{x=0\}$ by D_1, D_2, D_3, D_4 and D_5 (or sometimes D_0) respectively. It is because $c^i(D_0)=D_i$. These five divisors form a pentagon and we denote each vertex $D_i\cap D_{i-1}$ by P_i. Hence we have $c^i(P_0)=P_i$. The two dimensional affine space $U_1=Spec\mathbf{Q}[x,y]$ gives an open affine subset of $\overline{\mathcal{M}_{0,5}}$. The S_5-action gives other open subsets $U_i=c^{i-1}(U_1)=Spec\mathbf{Q}[z_i,w_i]$ $(1\leqslant i\leqslant 5)$ in $\overline{\mathcal{M}_{0,5}}$ where $(z_1,w_1)=(x,y)$, $(z_2,w_2)=(y,\frac{1-x}{1-xy})$, $(z_3,w_3)=(\frac{1-x}{1-xy},1-xy)$, $(z_4,w_4)=(1-xy,\frac{1-y}{1-xy})$ and $(z_5,w_5)=(\frac{1-y}{1-xy},x)$.

For $\mathbf{a}=(a_1,\cdots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\cdots,b_l)\in\mathbf{Z}^l_{>0}$, and x, $y\in\mathbf{Q}_p$ with $|x|_p<1$ and $|y|_p<1$ we define **two variable p-adic multiple polylogarithm** by

$$\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x,y):=\sum_{\substack{0<m_1<\cdots<m_k\\<n_1<\cdots<n_l}}\frac{x^{m_k}\qquad y^{n_l}}{m_1^{a_1}\cdots m_k^{a_k}n_1^{b_1}\cdots n_l^{b_l}}\in\mathbf{Q}_p[[x,y]],$$

and for $\mathbf{c}=(c_1,\cdots,c_h)\in\mathbf{Z}^h_{>0}$ **one variable p-adic multiple polylogarithm** by

$$\mathrm{Li}_{\mathbf{c}}(y):=\sum_{0<m_1<\cdots<m_h}\frac{y^{m_h}}{m_1^{c_1}\cdots m_h^{c_h}}\in\mathbf{Q}_p[[y]]\subset\mathbf{Q}_p[[x,y]].$$

It is easy to see that these functions satisfy the following differential equations.

(5.2)

$$\frac{d}{dx}\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x,y)=$$

$$\begin{cases}\frac{1}{x}\mathrm{Li}_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(x,y) & \text{if } a_k\neq 1,\\ \frac{1}{1-x}\mathrm{Li}_{(a_1,\cdots,a_{k-1}),\mathbf{b}}(x,y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)\mathrm{Li}_{(a_1,\cdots,a_{k-1},b_1),(b_2,\cdots,b_l)}(x,y) & \\ & \text{if } a_k=1,k\neq 1,l\neq 1,\\ \frac{1}{1-x}\mathrm{Li}_{\mathbf{b}}(y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)\mathrm{Li}_{(b_1),(b_2,\cdots,b_l)}(x,y) & \text{if } a_k=1,k=1,l\neq 1,\\ \frac{1}{1-x}\mathrm{Li}_{(a_1,\cdots,a_{k-1}),\mathbf{b}}(x,y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)\mathrm{Li}_{(a_1,\cdots,a_{k-1},b_1)}(xy) & \\ & \text{if } a_k=1,k\neq 1,l=1,\\ \frac{1}{1-x}\mathrm{Li}_{\mathbf{b}}(y)-\left(\frac{1}{x}+\frac{1}{1-x}\right)\mathrm{Li}_{\mathbf{b}}(xy) & \text{if } a_k=1,k=1,l=1,\end{cases}$$

$$\frac{d}{dy}\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x,y)=\begin{cases}\frac{1}{y}\mathrm{Li}_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(x,y) & \text{if } b_l\neq 1,\\ \frac{1}{1-y}\mathrm{Li}_{\mathbf{a},(b_1,\cdots,b_{l-1})}(x,y) & \text{if } b_l=1,l\neq 1,\\ \frac{1}{1-y}\mathrm{Li}_{\mathbf{a}}(xy) & \text{if } b_l=1,l=1,\end{cases}$$

$$\frac{d}{dx}\mathrm{Li}_{\mathbf{c}}(xy)=\begin{cases}\frac{1}{x}\mathrm{Li}_{(c_1,\cdots,c_{h-1},c_h-1)}(xy) & \text{if } c_h\neq 1,\\ \frac{y}{1-xy}\mathrm{Li}_{(c_1,\cdots,c_{h-1})}(xy) & \text{if } c_h=1,h\neq 1,\\ \frac{y}{1-xy} & \text{if } c_h=1,h=1,\end{cases}\tag{5.3}$$

$$\frac{d}{dy}\mathrm{Li}_{\mathbf{c}}(xy)=\begin{cases}\frac{1}{y}\mathrm{Li}_{(c_1,\cdots,c_{h-1},c_h-1)}(xy) & \text{if } c_h\neq 1,\\ \frac{x}{1-xy}\mathrm{Li}_{(c_1,\cdots,c_{h-1})}(xy) & \text{if } c_h=1,h\neq 1,\\ \frac{x}{1-xy} & \text{if } c_h=1,h=1,\end{cases}$$

$$\frac{d}{dx}\mathrm{Li}_{\mathbf{c}}(y)=0\tag{5.4}$$

$$\frac{d}{dy}\mathrm{Li}_{\mathbf{c}}(y)=\begin{cases}\frac{1}{y}\mathrm{Li}_{(c_1,\cdots,c_{h-1},c_h-1)}(y) & \text{if } c_h\neq 1,\\ \frac{1}{1-y}\mathrm{Li}_{(c_1,\cdots,c_{h-1})}(y) & \text{if } c_h=1,h\neq 1,\\ \frac{1}{1-y} & \text{if } c_h=1,h=1,\end{cases}$$

By the above differential equations, $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ are all iterated integrals of $\frac{dx}{x}$, $\frac{dx}{1-x}$, $\frac{dy}{y}$, $\frac{dy}{1-y}$ and $\frac{xdy+ydx}{1-xy}$, differential forms over $\mathcal{M}_{0,5}$. Whence they are obtained from some triple (M,s,y) over $\mathcal{M}_{0,5}$. We interpret them as Coleman functions over the rigid triple $(\mathcal{M}_{0,5},\overline{\mathcal{M}_{0,5}})$. This means that they are analytically continued to $\mathcal{M}_{0,5}(\mathbf{Q}_p)$ as Coleman functions by the methods of analytically continuation along Frobenius in §2.

6. Analytic continuation of two variable p-adic multiple polylogarithms

In this section two (and one) variable p-adic multiple polylogarithms are analytically continued into $N^{00}_{D_i}$ $(i \in \mathbf{Z}/5)$, which is a Zariski open subset of the normal bundle N_{D_i} of the divisor D_i introduced in the previous section.

Notation 6.1. For a Coleman function f over $\mathcal{M}_{0,5}$, $f^{(D_i)}$ means the analytic continuation of f to $N^{00}_{D_i}$ $(i \in \mathbf{Z}/5)$. For $\mathbf{a}=(a_1,\cdots,a_k)\in \mathbf{Z}^k_{>0}$ and $\mathbf{b}=(b_1,\cdots,b_l)\in \mathbf{Z}^l_{>0}$, $F_{\mathbf{a},\mathbf{b}}$ stands for the Coleman function $Li_{\mathbf{a},\mathbf{b}}(x,y)-Li_{\mathbf{ab}}(xy)$ and for $\mathbf{c}=(c_1,\cdots,c_h)\in\mathbf{Z}^h_{>0}$, $G_{\mathbf{c}}$ stands for the Coleman function $Li_{\mathbf{c}}(xy)-Li_{\mathbf{c}}(y)$ over $\mathcal{M}_{0,5}$.

Lemma 6.2. *$F^{(D_1)}_{\mathbf{a},\mathbf{b}}=0$ and $G^{(D_1)}_{\mathbf{c}}=0$ for any index* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$.

Proof . The constant terms of $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ at the origin P_5 are zero because there are no constant terms in their power series expansions. We take their differentials and take their residues at $y=0$. It gives 0 by induction because each term will be a multiple polylogarithm with one lower weight than the original one. Whence $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ are identically zero. It gives our claim. □

Lemma 6.3. *$F^{(D_2)}_{\mathbf{a},\mathbf{b}}\equiv 0$ if* $\mathbf{a}$ *is admissible* [1] *and $G^{(D_2)}_{\mathbf{c}}\equiv 0$ for any index* $\mathbf{c}$.

Proof . On the affine coordinate (z_2,w_2) for U_2, the divisor D_2 is defined by $w_2=0$. By using $dx=\frac{w_2(1-w_2)}{(z_2w_2-1)^2}dz_2+\frac{z_2-1}{(z_2w_2-1)^2}dw_2$ and $dy=dz_2$, we obtain the following from the differential equations in

[1] An index $\mathbf{a}=(a_1,\cdots,a_k)$ $(a_i\in\mathbf{N})$ is called *admissible* if $a_k>1$.

(5.2) $\sim$ (5.4).

$$\frac{d}{dz_2}Li_{\mathbf{a},\mathbf{b}}(x,y)=\begin{cases}\frac{w_2}{1-z_2w_2}Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(x,y)+\frac{1}{z_2}Li_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(x,y) & \\ \qquad\qquad\qquad \text{if } a_k>1, b_l\neq 1, \\ \frac{w_2}{1-z_2w_2}Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(x,y)+\frac{1}{1-z_2}Li_{\mathbf{a},(b_1,\cdots,b_{l-1})}(x,y) & \\ \qquad\qquad\qquad \text{if } a_k>1, b_l=1, l\neq 1, \\ \frac{w_2}{1-z_2w_2}Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(x,y)+\frac{1}{1-z_2}Li_{\mathbf{a}}(xy) & \\ \qquad\qquad\qquad \text{if } a_k>1, b_l=1, l=1,\end{cases}$$

$$\frac{d}{dz_2}Li_{\mathbf{c}}(xy)=\begin{cases}\left(\frac{w_2}{1-z_2w_2}+\frac{1}{z_2}\right)\cdot Li_{(c_1,\cdots,c_{h-1},c_h-1)}(xy) & \text{if } c_h\neq 1,\\ \left(\frac{z_2w_2(1-w_2)}{(z_2-1)(z_2w_2-1)}+\frac{w_2-1}{z_2-1}\right)\cdot Li_{(c_1,\cdots,c_{h-1})}(xy) & \text{if } c_h=1, h\neq 1,\\ \frac{z_2w_2(1-w_2)}{(z_2-1)(z_2w_2-1)}+\frac{w_2-1}{z_2-1} & \text{if } c_h=1, h=1,\end{cases}$$

$$\frac{d}{dz_2}Li_{\mathbf{c}}(y)=\begin{cases}\frac{1}{z_2}Li_{(c_1,\cdots,c_{h-1},c_h-1)}(y) & \text{if } c_h\neq 1,\\ \frac{1}{1-z_2}Li_{(c_1,\cdots,c_{h-1})}(y) & \text{if } c_h=1, h\neq 1,\\ \frac{1}{1-z_2} & \text{if } c_h=1, h=1.\end{cases}$$

Following Proposition 4.4, we compute the residue around D_2

$$\frac{d}{d\bar z_2}Li^{(D_2)}_{\mathbf{a},\mathbf{b}}(x,y)=\begin{cases}\frac{1}{\bar z_2}Li^{(D_2)}_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(x,y) & \text{if } a_k>1, b_l\neq 1,\\ \frac{1}{1-\bar z_2}Li^{(D_2)}_{\mathbf{a},(b_1,\cdots,b_{l-1})}(x,y) & \text{if } a_k>1, b_l=1, l\neq 1,\\ \frac{1}{1-\bar z_2}Li^{(D_2)}_{\mathbf{a}}(xy) & \text{if } a_k>1, b_l=1, l=1,\end{cases}$$

$$\frac{d}{d\bar z_2}Li^{(D_2)}_{\mathbf{c}}(xy)=\begin{cases}\frac{1}{\bar z_2}Li^{(D_2)}_{(c_1,\cdots,c_{h-1},c_h-1)}(xy) & \text{if } c_h\neq 1,\\ \frac{1}{1-\bar z_2}Li^{(D_2)}_{(c_1,\cdots,c_{h-1})}(xy) & \text{if } c_h=1, h\neq 1,\\ \frac{1}{1-\bar z_2} & \text{if } c_h=1, h=1,\end{cases}$$

$$\frac{d}{d\bar z_2}Li^{(D_2)}_{\mathbf{c}}(y)=\begin{cases}\frac{1}{\bar z_2}Li^{(D_2)}_{(c_1,\cdots,c_{h-1},c_h-1)}(y) & \text{if } c_h\neq 1,\\ \frac{1}{1-\bar z_2}Li^{(D_2)}_{(c_1,\cdots,c_{h-1})}(y) & \text{if } c_h=1, h\neq 1,\\ \frac{1}{1-\bar z_2} & \text{if } c_h=1, h=1.\end{cases}$$

Similarly we compute

$$\begin{aligned}
&\frac{d}{d\bar{w}_2} Li^{(D_2)}_{\mathbf{a},\mathbf{b}}(x,y) = 0 && \text{if } a_k > 1,\\
&\frac{d}{d\bar{w}_2} Li^{(D_2)}_{\mathbf{c}}(xy) = 0, &&\\
&\frac{d}{d\bar{w}_2} Li^{(D_2)}_{\mathbf{c}}(y) = 0. &&
\end{aligned}$$

These computations imply that differentials of $F^{(D_2)}_{\mathbf{ab}}$ and $G^{(D_2)}_{\mathbf{c}}$ with respect to $\bar{z}_2$ and $\bar{w}_2$ are zero by induction. Therefore they must be constant. By Lemma 6.2 their constant terms at P_1 is zero. So they are identically zero. $\square$

Lemma 6.4. $F^{(D_3)}_{\mathbf{a},\mathbf{b}} = 0$ *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible and* $G^{(D_3)}_{\mathbf{c}} = 0$ *if* $\mathbf{c}$ *is admissible.*

Proof . On the affine coordinate (z_3, w_3) for U_3, the divisor D_3 is defined by $w_3 = 0$. By using $dx = -w_3 dz_3 - z_3 dw_3$ and $dy = \frac{w_3(1-w_3)}{(z_3w_3-1)^2} dz_3 + \frac{z_3-1}{(z_3w_3-1)^2} dw_3$, we obtain the following from the differential equations in $(5.2) \sim (5.4)$.

$$\begin{aligned}
&\frac{d}{dz_3} Li_{\mathbf{a},\mathbf{b}}(x,y) = \frac{w_3}{z_3w_3-1} Li_{(a_1,\cdots,a_{k-1},a_k-1),\mathbf{b}}(x,y)\\
&\qquad + \frac{w_3}{1-z_3w_3} Li_{\mathbf{a},(b_1,\cdots,b_{l-1},b_l-1)}(x,y) && \text{if } a_k > 1 \text{ and } b_l > 1,\\
&\frac{d}{dz_3} Li_{\mathbf{c}}(xy) = 0 && \text{if } c_h > 1,\\
&\frac{d}{dz_3} Li_{\mathbf{c}}(y) = \frac{w_3}{1-z_3w_3} Li_{(c_1,\cdots,c_{h-1},c_h-1)}(y) && \text{if } c_h > 1.
\end{aligned}$$

Following Proposition 4.4, we compute the residue around D_3

$$\begin{aligned}
&\frac{d}{d\bar{z}_3} Li^{(D_3)}_{\mathbf{a},\mathbf{b}}(x,y) = 0 && \text{if } a_k > 1 \text{ and } b_l > 1,\\
&\frac{d}{d\bar{z}_3} Li^{(D_3)}_{\mathbf{c}}(xy) = 0 && \text{if } c_h > 1,\\
&\frac{d}{d\bar{z}_3} Li^{(D_3)}_{\mathbf{c}}(y) = 0 && \text{if } c_h > 1.
\end{aligned}$$

Similarly we also compute

$$\frac{d}{d\bar{w}_3}Li^{(D_3)}_{\mathbf{a},\mathbf{b}}(x,y)=0 \quad \text{if } a_k>1 \text{ and } b_l>1,$$
$$\frac{d}{d\bar{w}_3}Li^{(D_3)}_{\mathbf{c}}(xy)=0 \quad \text{if } c_h>1,$$
$$\frac{d}{d\bar{w}_3}Li^{(D_3)}_{\mathbf{c}}(y)=0 \quad \text{if } c_h>1.$$

These computations imply that differentials of $F^{(D_3)}_{\mathbf{a},\mathbf{b}}$ and $G^{(D_3)}_{\mathbf{c}}$ with respect to $\bar{z}_3$ and $\bar{w}_3$ are zero by induction. Therefore they must be constant. By Lemma 6.3 their constant term at P_2 is zero. So they are identically zero. □

In [F1] it was shown that the limit (in a certain way) to $z=1$ of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$, which is a Coleman function over $\mathbf{P}^1\backslash\{0,1,\infty\}$, exists when $k_m>1$ (loc. cit. Theorem 2.18) and p-adic multiple zeta value $\zeta_p(k_1,\cdots,k_m)$ is defined to be this limit value (loc. cit. Definition 2.17), but by using the terminologies in §3 we reformulate its definition as follows.

Definition 6.5. For $k_m>1$, the **p-adic multiple zeta value** $\zeta_p(k_1,\cdots,k_m)$ is the constant term of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$ at $z=1$.

In the case for $k_m=1$, the constant term of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$ at $z=1$ is actually equal to the (canonical) regularization $(-1)^m I_p(BA^{k_{m-1}-1}B\cdots A^{k_1-1}B)$ of p-adic multiple zeta values by loc. cit. Theorem 2.22 (for this notation, see loc. cit. Theorem 3.30).

The following is important to prove double shuffle relations for p-adic multiple zeta values.

Proposition 6.6. (1) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(x,y)$ *is constant and equal to* $\zeta_p(\mathbf{a},\mathbf{b})$ *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible.*
(2) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(xy)$ *and* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(y)$ *are constant and take value* $\zeta_p(\mathbf{c})$ *if* $\mathbf{c}$ *is admissible.*

Proof . By Lemma 6.4 it is enough to prove this for $Li^{(D_3)}_{\mathbf{c}}(y)$. By the argument in Lemma 6.2 $Li^{(D_1)}_{\mathbf{c}}(y)=0$. By the computation in Lemma 6.3 $Li^{(D_2)}_{\mathbf{c}}(y)=Li^{(D_2)}_{\mathbf{c}}(\bar{z}_2)$. So the constant term of $Li^{(D_2)}_{\mathbf{c}}(y)$ at P_2 is equal to the constant term of $Li_{\mathbf{c}}(\bar{z}_2)$ at $\bar{z}_2=1$, which is $\zeta_p(\mathbf{c})$. By the computation in Lemma 6.4 $Li^{(D_3)}_{\mathbf{c}}(y)$ must be constant if $c_h>1$. Since this constant term must be the constant term of $Li^{(D_2)}_{\mathbf{c}}(y)$, $Li^{(D_3)}_{\mathbf{c}}(y)\equiv\zeta_p(\mathbf{c})$ for $c_h>1$. □

By discussing on the opposite divisors D_5, D_4 and D_3, we also obtain the following.

Proposition 6.7. (1) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(y,x)$ *is constant and equal to* $\zeta_p(\mathbf{a},\mathbf{b})$ *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible.*

(2) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(xy)$ *and* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(x)$ *are constant and equal to* $\zeta_p(\mathbf{c})$ *if* $\mathbf{c}$ *is admissible.*

7. The double shuffle relations

In this section, we prove double shuffle relations for p-adic multiple zeta values (Definition 6.5). Firstly we recall double shuffle relations for complex multiple zeta values. Let $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{b} = (b_1, \cdots, b_l)$ be admissible indices (i.e. $a_k > 1$ and $b_l > 1$). The **series shuffle product formulas** (called by harmonic product formulas in [F1] and first shuffle relations in [G1]) are relations

$$\zeta(\mathbf{a}) \cdot \zeta(\mathbf{b}) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} \zeta(\sigma(\mathbf{a},\mathbf{b})) \tag{7.1}$$

which is obtained by expanding the summation on the left hand side into the summation which give multiple zeta values. Here

$$Sh^{\leqslant}(k,l) := \bigcup_N \Big\{ \sigma : \{1, \cdots, k+l\} \to \{1, \cdots, N\} \Big| \ \sigma \text{ is onto}, \\ \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l) \Big\}$$

and $\sigma(\mathbf{a},\mathbf{b}) = (c_1, \cdots, c_N)$ where N is the cardinality of the image of σ and

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k. \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k. \end{cases}$$

One of the easiest example of (7.1) is (0.2).

On the other hand, multiple zeta values admit an iterated integral expression (cf. [G1], [IKZ] see also [F0])

$$\zeta(\mathbf{a}) = \int_0^1 \underbrace{\frac{du}{u} \circ \cdots \circ \frac{du}{u} \circ \frac{du}{1-u}}_{a_k} \circ \frac{du}{u} \circ \cdots\cdots \circ \frac{du}{1-u} \\ \circ \underbrace{\frac{du}{u} \circ \cdots \circ \frac{du}{u} \circ \frac{du}{1-u}}_{a_1}.$$

Here for differential 1-forms $\omega_1,\omega_2,\dots,\omega_n$ ($n\geqslant 1$) on $\mathbf{C}$ an iterated integral $\int_0^1\omega_1\circ\omega_2\circ\cdots\circ\omega_n$ is defined inductively as $\int_0^1\omega_1(t_1)\int_0^{t_1}\omega_2\circ\cdots\circ\omega_n$. There are the well-known shuffle product formulas (for example see loc. cit.) of iterated integration

$$\int_0^1\omega_1\circ\cdots\circ\omega_k\cdot\int_0^1\omega_{k+1}\circ\cdots\circ\omega_{k+l}=\sum_{\tau\in Sh(k,l)}\int_0^1\omega_{\tau(1)}\circ\cdots\circ\omega_{\tau(k+l)},$$

where $Sh(k,l)$ is the set of shuffles defined by

$$\begin{aligned}Sh(k,l):=\Big\{\tau:&\{1,\cdots,k+l\}\to\{1,\cdots,k+l\}\Big|\ \tau\text{ is bijective,}\\ &\tau(1)<\cdots<\tau(k),\tau(k+1)<\cdots<\tau(k+l)\Big\}.\end{aligned}$$

They induce the **iterated integral shuffle produce formulas** (called by shuffle product formulas simply in [F1] and second shuffle relations in [G1]) for multiple zeta values

$$\zeta(\mathbf{a})\cdot\zeta(\mathbf{b})=\sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})}\zeta(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})})\tag{7.2}$$

where $N_{\mathbf{a}}=a_1+\cdots+a_k$, $N_{\mathbf{b}}=b_1+\cdots+b_l$. For $\mathbf{c}=(c_1,\cdots,c_h)$ with $h,c_1,\dots,c_h\geqslant 1$ the symbol $W_{\mathbf{c}}$ means a word $A^{c_h-1}BA^{c_{h-1}-1}B\cdots A^{c_1-1}B$ and conversely for given such W we denote its corresponding index by I_W. For words, $W=X_1\cdots X_k$ and $W'=X_{k+1}\cdots X_{k+l}$ with $X_i\in\{A,B\}$, and $\tau\in Sh(k,l)$ the symbol $\tau(W,W')$ stands for the word $Z_1\cdots Z_{k+l}$ with $Z_i=X_{\tau^{-1}(i)}$. One of the easiest example of (7.2) is (0.4).

The **double shuffle relations** for multiple zeta values are linear relations which are obtained by combining two shuffle relations (7.3), i.e. series shuffle product formulas (7.1) and iterated integral shuffle produce formulas (7.2)

$$\sum_{\sigma\in Sh^{\leqslant}(k,l)}\zeta(\sigma(\mathbf{a},\mathbf{b}))=\sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})}\zeta(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})}).\tag{7.3}$$

The following is the easiest example of the double shuffle relations obtained from (0.2) and (0.4):

$$\begin{aligned}&\zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\\ &=\sum_{i=0}^{k_1-1}\binom{k_2-1+i}{i}\zeta(k_1-i,k_2+i)+\sum_{j=0}^{k_2-1}\binom{k_1-1+j}{j}\zeta(k_2-j,k_1+j)\end{aligned}$$

for $k_1,k_2>1$.

Theorem 7.1. *p-adic multiple zeta values in convergent case (i.e. for admissible indices) satisfy the series shuffle product formulas, i.e.*

$$\zeta_p(\mathbf{a}) \cdot \zeta_p(\mathbf{b}) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} \zeta_p(\sigma(\mathbf{a}, \mathbf{b})) \tag{7.4}$$

for admissible indices $\mathbf{a}$ *and* $\mathbf{b}$*.*

Proof . Put $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{b} = (b_1, \cdots, b_l)$. By the power series expansion of $\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x, y)$ and $\mathrm{Li}_{\mathbf{a}}(x)$ in §5, we obtain the following formula

$$\mathrm{Li}_{\mathbf{a}}(x) \cdot \mathrm{Li}_{\mathbf{b}}(y) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} \mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x, y). \tag{7.5}$$

Here

$$\mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x, y) := \sum_{(m_1,\cdots,m_k,n_1,\cdots,n_l) \in Z^{\sigma}_{++}} \frac{x^{m_k} \qquad y^{n_l}}{m_1^{a_1} \cdots m_k^{a_k} n_1^{b_1} \cdots n_l^{b_l}}$$

with

$$Z^{\sigma}_{++} = \left\{ (c_1, \cdots, c_{k+l}) \in \mathbf{Z}^{k+l}_{>0} \;\middle|\; c_i < c_j \text{ if } \sigma(i) < \sigma(j), c_i = c_j \text{ if } \sigma(i) = \sigma(j) \right\}.$$

Then for each $\sigma \in Sh^{\leqslant}(k, l)$, $\mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x, y)$ can be written $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(x, y)$, $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(y, x)$ or $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(xy)$ for some indices $\mathbf{a}'$ and $\mathbf{b}'$. We note that, if $\mathbf{a}$ and $\mathbf{b}$ are admissible, then these $\mathbf{a}'$ and $\mathbf{b}'$ are also admissible. By Proposition 6.6 and Proposition 6.7, we know that analytic continuations $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(x, y)$, $\mathrm{Li}^{(D_3)}_{\mathbf{b},\mathbf{a}}(y, x)$, $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(xy)$, $\mathrm{Li}^{(D_3)}_{\mathbf{a}}(x)$ and $\mathrm{Li}^{(D_3)}_{\mathbf{b}}(y)$ are all constant and take values $\zeta_p(\mathbf{a}, \mathbf{b})$, $\zeta_p(\mathbf{b}, \mathbf{a})$, $\zeta_p(\mathbf{a}, \mathbf{b})$, $\zeta_p(\mathbf{a})$ and $\zeta_p(\mathbf{b})$ respectively when $\mathbf{a}$ and $\mathbf{b}$ are admissible. Therefore by taking an analytic continuation along Frobenius of both hands sides of (7.5) into $N^{00}_{D_3}(\mathbf{Q}_p)$, we obtain the series shuffle product formulas (7.4) for p-adic multiple zeta value in convergent case. □

By this theorem we say for example

$$\zeta_p(k_1) \cdot \zeta_p(k_2) = \zeta_p(k_1, k_2) + \zeta_p(k_2, k_1) + \zeta_p(k_1 + k_2)$$

for $k_1, k_2 > 1$ which is a p-adic analogue of (0.2).

Corollary 7.2. *p-adic multiple zeta values in convergent case satisfy double shuffle relations. Namely*

$$\sum_{\sigma \in Sh^{\leqslant}(k,l)} \zeta_p(\sigma(\mathbf{a}, \mathbf{b})) = \sum_{\tau \in Sh(N_{\mathbf{a}}, N_{\mathbf{b}})} \zeta_p(I_{\tau(W_{\mathbf{a}}, W_{\mathbf{b}})}).$$

holds for $a_k > 1$ *and* $b_l > 1$*.*

Proof . It was shown in [F1] Corollary 3.46 that p-adic multiple zeta values satisfy iterated integral shuffle product formulas

$$\zeta_p(\mathbf{a})\cdot\zeta_p(\mathbf{b})=\sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})}\zeta_p(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})}). \tag{7.6}$$

By combining it with Theorem 7.1, we obtain double shuffle relations for p-adic multiple zeta values. □

Therefore we say for example

$$\zeta_p(k_1,k_2)+\zeta_p(k_2,k_1)+\zeta_p(k_1+k_2)$$
$$=\sum_{i=0}^{k_1-1}\binom{k_2-1+i}{i}\zeta_p(k_1-i,k_2+i)+\sum_{j=0}^{k_2-1}\binom{k_1-1+j}{j}\zeta_p(k_2-j,k_1+j)$$

for $k_1,k_2>1$ which is a p-adic analogue of (0.4).

In complex case there are two regularizations of multiple zeta values in divergent case, integral regularization and power series regularization (see [IKZ], [G1]§2.9 and §2.10). The first ones satisfy iterated integral shuffle product formulas, the second ones satisfy series shuffle product formulas and these two regularizations are related by regularization relations. Actually these provide new type of relations among multiple zeta values. In the case of p-adic multiple zeta values, p-adic analogue of integral regularization appear on coefficients of p-adic Drinfel'd associator (see [F1]) and they satisfy iterated integral shuffle product formulas like (7.6). On the other hand, it is not clear at all to say that p-adic analogue of power series regularization satisfy series shuffle product formulas and regularization relation. It is because that in the complex case the definition of this regularization and the proof of their series shuffle product formulas and regularization relation essentially based on the asymptotic behaviors of power series summations of multiple zeta values (see [G1] Proposition 2.19) however in the p-adic case our p-adic multiple zeta values do not have power series sum expression like (0.1). Recently the validity of these type of relations among p-adic multiple zeta values were achieved in [FJ] by using several variable p-adic multiple polylogarithm and a stratification of the moduli $\mathcal{M}_{0,N+3}$ $(N\geqslant 3)$.

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